# Chapter 13
# Mechanics of non viscous fluids

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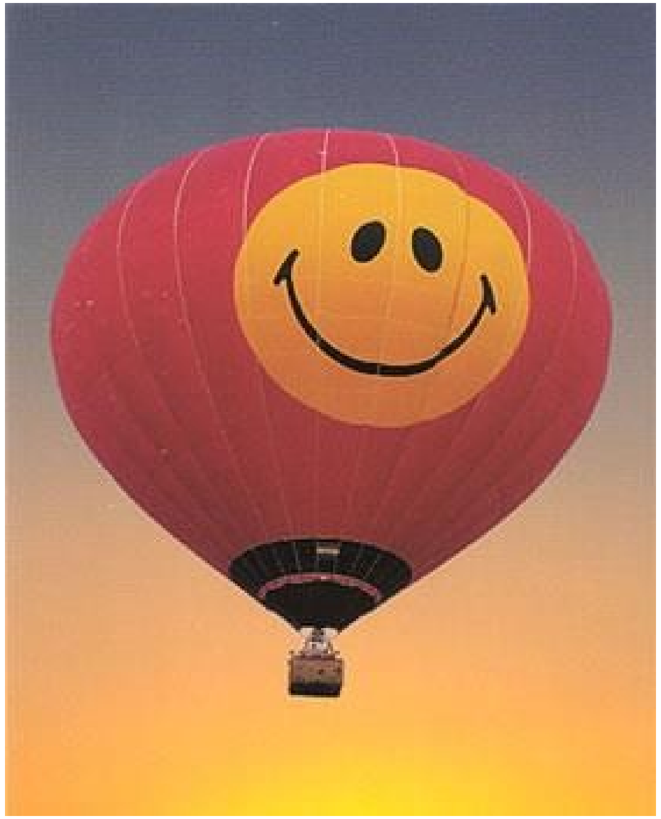
# COURSE TOPICS:

13.1 Archimedes' Principle
13.2 The equation of continuity, Streamline flow
13.3 Bernoulli's Equation
13.4 Static consequence of Bernoulli's equation

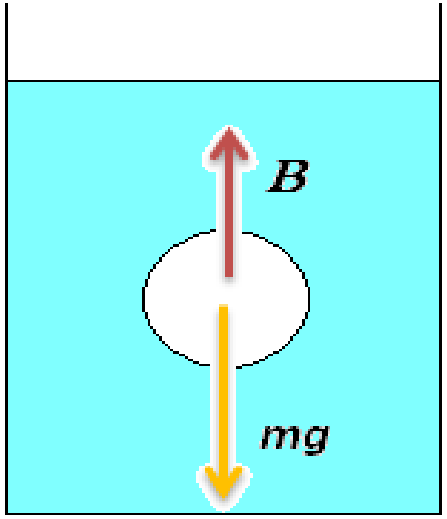


# Introduction:

- In this chapter we discuss fluids at rest and **non-viscous (frictionless)** fluids motion.
- We first develop an understanding of why an object may either sink or float in a fluid at rest? (Archimedes' principle).
- We then develop the Bernoulli's Equation, which puts work and energy concepts into a form suitable for fluids.
- Then, we can understand why fluids in connected containers tend to have the same surface levels? And how fluids ***flow*** from one place to another?
- On this discussion, the important condition is the assumption that the fluid is ***incompressible***:
  - **Incompressible fluid: a given mass of fluid always occupies the same volume though its shape may change.**
- In fluid mechanics, as a given mass of fluid does not have a fixed shape, the density and pressure are commonly used instead of mass and force.
- Recall that :
  - **The density is the mass per unit volume:** $\boldsymbol{\rho = m/V}$
  - **The pressure is the force per unit area:** $\vec{\boldsymbol{p}} = \vec{\boldsymbol{F}}/\boldsymbol{A}$

# 13.1: Archimedes' principle

- An object floating or submerged in a fluid experiences an upward or *Buoyant force* due to the fluid.
- **Archimedes' principle states that the buoyant force $B$ on the object is equal to the weight of the displaced fluid:**




- Consider a solid of volume $V$ and density $\rho$ <u>completely </u>submerged in a fluid of density $\rho_0$.
- The displaced fluid by the solid has a mass

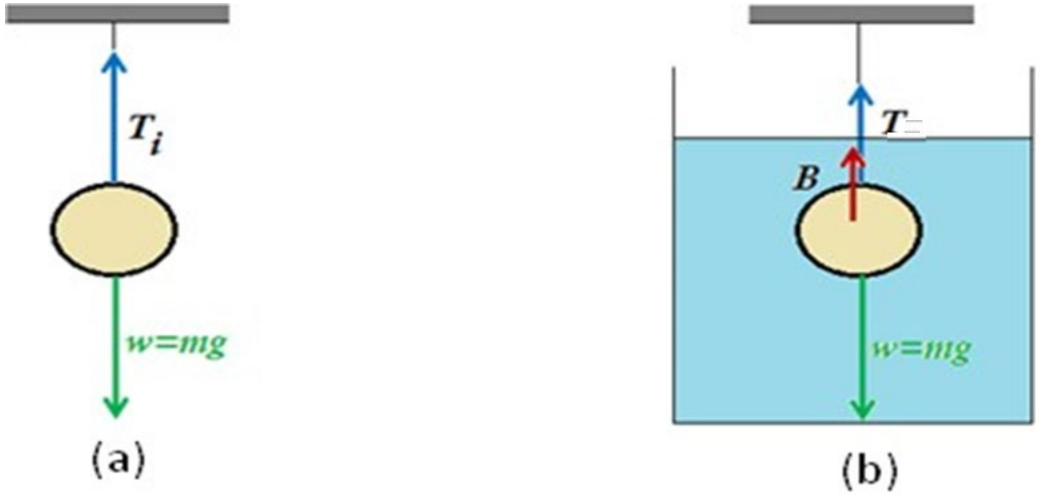
$$m_D = \rho_0 V_D$$
$$w_D = m_D g = \rho_0 V_D g$$

- Only gravity and the surrounding fluid exert forces on the segment.
- Since it remains at rest, the force, **B**, exerted by the remainder of the fluid must balance the weight
- The buoyant force (which is the weight of the displaced fluid) is then:

$$\boxed{B = \rho_0 V_D g}$$

- The buoyant force is the resultant force exerted by fluid on the surface of a submerged solid.

# 13.1: Archimedes' principle




- Suppose now that this imaginary segment of fluid is replaced by a heavier object of volume $V$ suspended by a string in the fluid.
  The surrounding fluid does not distinguish between the object and the fluid it replaces, so the buoyant force is the same.

- **Thus, the buoyant force on the object is equal to the weight of the displaced fluid:**

$$\boxed{B = \rho_o V_D\, g}$$

- The tension in the string is decreased when the object is placed in the fluid.
  - The density of the suspended object is $\rho$, and its weight is $w = \rho g V$.
  - The upward forces are the tension $\boldsymbol{T}$ and the buoyant force, $B = \rho_0 g V$.
  - Since it is equilibrium,

$$T + B = w$$

$$T = w - B = \rho g V - \rho_0 g V$$

$$\boxed{T = (\rho - \rho_0) g V}$$

- The tension in the string is reduced by the weight of the displaced fluid.
  - Archimedes' principle provides a way to determine densities

# 13.1: Archimedes' principle

**Example:** What is the magnitude of the buoyant force exerted on a piece of solid of volume $V$=15 $cm^3$ completely immersed in water.

**Answer**

$$B = \rho_0 V_D \mathrm{g} = \left(1000\,\frac{\mathrm{kg}}{\mathrm{m}^3}\right)(15 \times 10^{-6}\ m^3)\left(9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}\right) = 0.147N$$

# 13.1: Archimedes' principle

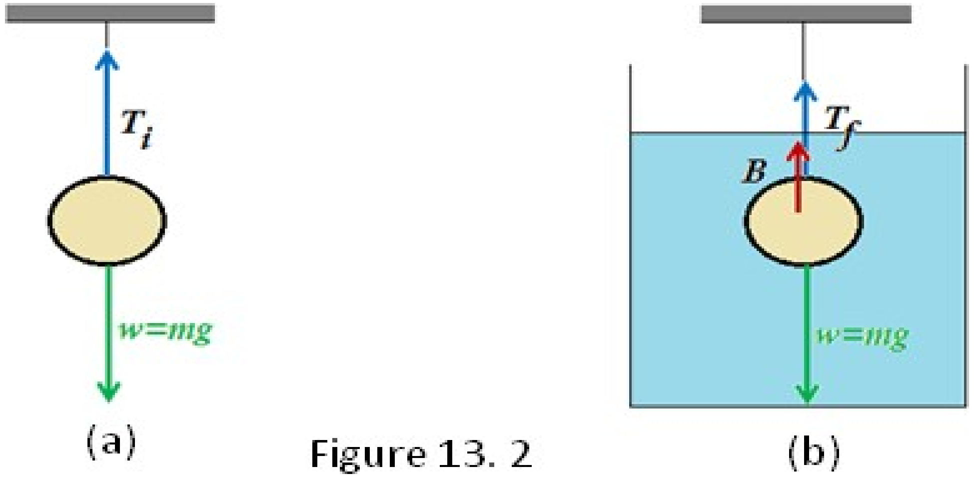


Figure 13. 2

**Example 13.1**: A piece of metal of unknown volume $V$ is suspended from a string. Before submersion, the tension in the string is $10\ N$. When the metal is submerged in water the tension is 8 N. The water density is $\rho_o = 10^3\ kg/m^3$. (a) calculate the buoyant force (b) calculate the volume of the piece of metal (c) What is the density of the metal?

**solution:**

a) Before submersion (Fig. 13.2 a) the tension of the cord is equal to the weight of the piece of metal:

$$T_i = w = mg = 10N$$

After submersion (Fig. 13.2 b), since the object is in equilibrium:

$$T_f + B - w = 0$$

Then the buoyant force is

$$B = w - T_f = T_i - T_f = 10N - 8N = 2N$$

b) The buoyant force is $B = \rho_o V_D g.$ The piece of metal is completely submerged in the fluid, then the volume of the displaced fluid is equal to the volume of the solid:

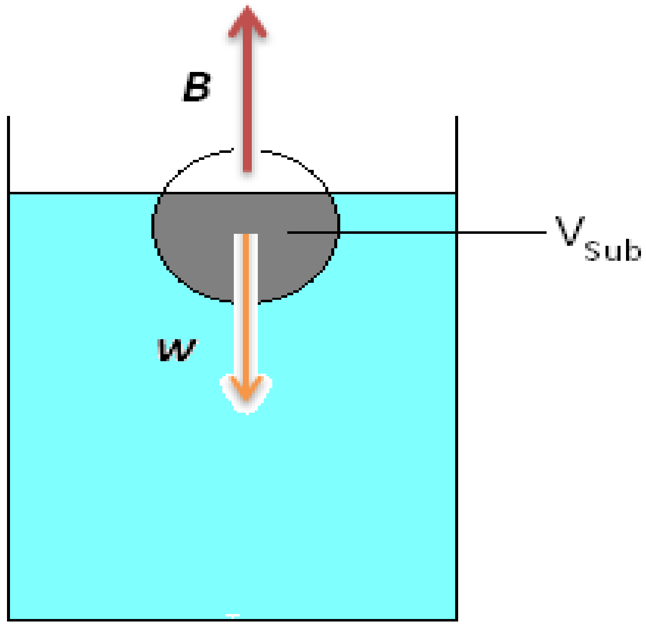
$$V = V_D = \frac{B}{\rho_0 g} = \frac{2\ N}{(1000 kg/m^3)(9.8\ m/s^2)} = 2.041 \times 10^{-4}\ m^3 = 204.1\ cm^3$$

c) Before submersion, the tension is $T_i = mg = \rho g V$

After submersion, the tension is $T_f = (\rho - \rho_0) g V$

Divide the two equations: $\frac{T_f}{T_i} = \frac{\rho - \rho_0}{\rho} \quad \Longrightarrow \quad \rho = \frac{\rho_0 T_i}{T_i - T_f} = \frac{(1000\ kg\ m^{-3})(10\ N)}{10\ N - 8\ N} = 5000\ kg/m^3$

# 13.1: Archimedes' principle
## Partially submerged




- If an object of volume $V$ is not completely immersed in a fluid, the displaced volume is equal to the submerged volume of the solid $V_s$ ( volume of the part of the solid below the top surface of the fluid) .
- Then a quantity without unit called *submerged fraction* is defined by the ratio of the submerged volume and the total volume of the solid: $\frac{V_S}{V}$
- By equating the buoyant force and the weight of the object

$$B = w,$$

$$\rho_o g V_s = \rho gV$$

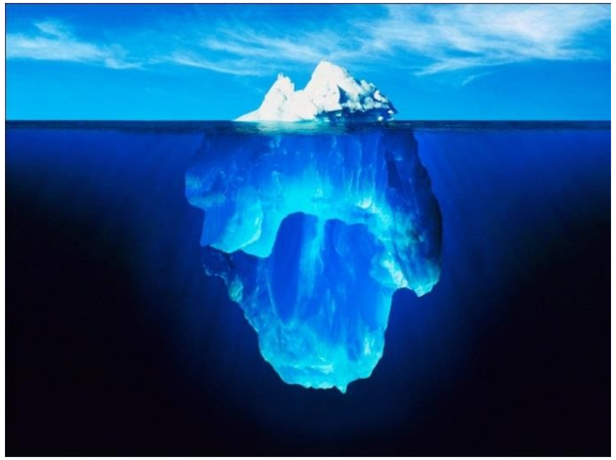
- The submerged fraction is then equal to the ratio of the density of

$$\boxed{\frac{\rho}{\rho_0} = \frac{V_s}{V}}$$

- Thus, the ratio of the densities is equal to the fraction of the volume submerged.

# 13.1: Archimedes' principle
## Partially submerged

**Example 13.2:** The density of ice is 920 $kg/m^3$ while that of sea water is 1025 $kg/m^3$. What fraction of an iceberg is submerged?



**Solution**:

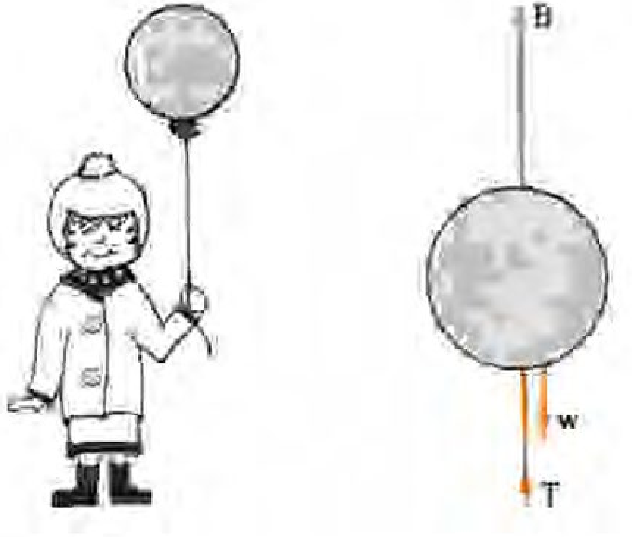
The fraction is

$$\frac{V_S}{V} = \frac{\rho}{\rho_0} = \frac{920\ \mathrm{kg\ m^{-3}}}{1025\ \mathrm{kg\ m^{-3}}} = 0.898$$

Almost 90 percent of the iceberg is submerged.

# 13.1: Archimedes' principle

## Partially submerged




**Example 13.3:** A child holds a helium-filled rubber balloon with a volume of 10 litres = 0.01 m$^3$ in air at 0 °C (Fig. 13.2a). Neglect the weight of the rubber and string and the buoyant force of the air on the child. (a) How great a force must she exert to keep the balloon from rising? (b) How many such balloons would it take to lift a 20-kg child?

**Solution:**

(a) According to Table 13.1, at 0 °C the density of helium is $0.178\ kg/m^3$, and the density of air is $1.29\ kg/m^3$. The weight of the helium $w = \rho_{He} gV$ is less than the upward buoyant force $B = \rho_{\text{air}} gV$, since air is denser.

Hence the child must pull down on the balloon with a force $\boldsymbol{T}$ to keep it from rising. If the balloon remains at rest,

$$B = T + w$$

$$\begin{aligned} T = B - w &= \rho_{\text{air}} gV - \rho_{\text{He}} gV \\ &= (1.29\ \text{kg m}^{-3} - 0.178\ \text{kg m}^{-3})(9.8\ \text{m s}^{-2})(0.01\ \text{m}^3) \\ &= 0.109\ \text{N} \end{aligned}$$

(b) Her weight is $w = mg = (20\ kg \times 9.8\ m\ s^{-2}) = 196\ N$.

Since each balloon can support 0.109 N, the number of balloons needed to balance her weight is

$$\frac{196}{0.109} = 1800 \text{ baloon}$$

# 13.2 The equation of continuity; Streamline flow

- **The flow rate $Q$** is the volume of the fluid flowing past a point in a channel per unit time:

$$\boxed{Q = \frac{\Delta V}{\Delta t}}$$

The S.I unit of the flow rate is cubic meters per second, $\boldsymbol{m^3/s}$.

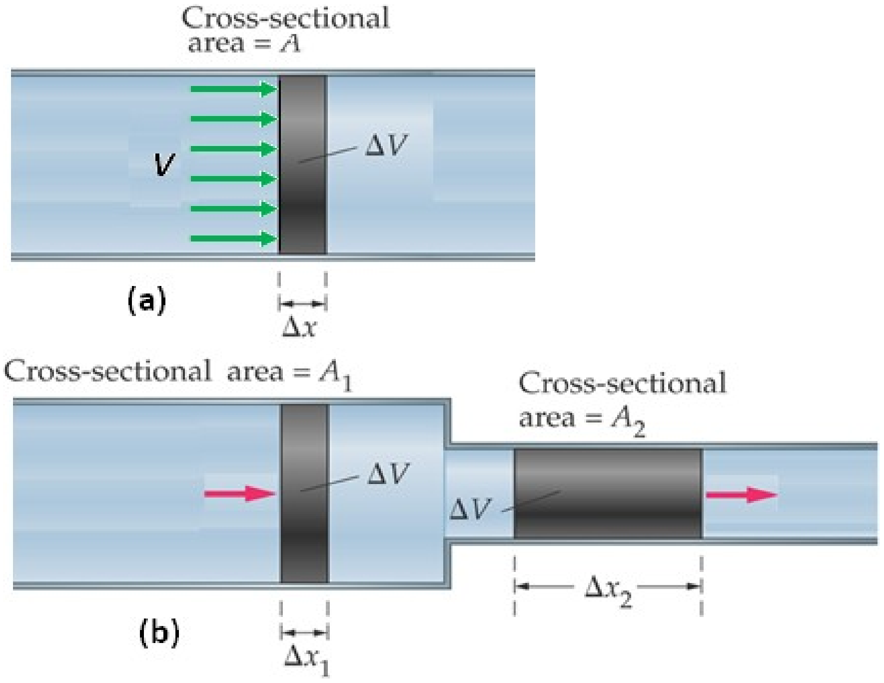
- For an **incompressible fluid** $(\rho = constant)$, the volume of fluid that passes any section of the tube per second is unchanged.

  - The fluid that enters one end of the channel such as a pipe or an artery at the flow rate $Q_1$, must leave the other end at a rate $Q_2$ which is the same.

  - Thus, **the equation of continuity** can be written as

$$\boxed{Q_1 = Q_2}$$

# 13.2 The equation of continuity; Streamline flow




- Consider a section of the tube with cross-sectional area $A$ and suppose that the fluid on this section has the same velocity $v$.
- In the time $\Delta t$ the fluid moves the distance

$$\Delta x = v\Delta t$$

- The volume of the fluid crossing the tube is

$$\Delta V = A\,\Delta x = Av\,\Delta t.$$

- The flow rate is then

$$\boxed{Q = \frac{\Delta V}{\Delta t} = Av}$$

- The flow rate equals the cross-sectional area of the channel times the velocity of the fluid.
- For a channel, whose cross-section changes from $A_1$ to $A_2$, this result together with $Q_1 = Q_2$ gives another form of the continuity equation:

$$\boxed{A_1 v_1 = A_2 v_2}$$

- The product of the cross-sectional area and the velocity of the fluid is constant

# 13.2 The equation of continuity; Streamline flow

**Example 13.4;**

A water pipe leading up to a hose has a radius of $1\ cm$. Water leaves the hose at a rate of 3 litres per minute.

1. Find the velocity of the water in the pipe.
2. The hose has a radius of $0.5\ cm$. What is the velocity of the water in the hose?

**Answer**

1. The velocity (strictly speaking, the average velocity) can be found from the flow rate and the area: $Q = Av$

The flow rate is the same in the hose and in the pipe.

Using $1\ litre = 10^{-3}\ m^3$ and $1\ \text{min} = 60\ s$, the flow rate is then:

$$Q = \frac{\Delta V}{\Delta t} = \frac{3\times10^{-3}m^3}{60s} = 5\times10^{-5}\ m^3/s$$

We will call the velocity and area in the pipe $v_1$ and $A_1$, respectively. Then, with $Q = A\ v$, we have:

$$v_1 = \frac{Q}{A_1} = \frac{Q}{\pi r_1^2} = \frac{5\times10^{-5}\ m^3\ s^{-1}}{\pi(0.01\ m)^2} = 0.159\ m^3/s$$

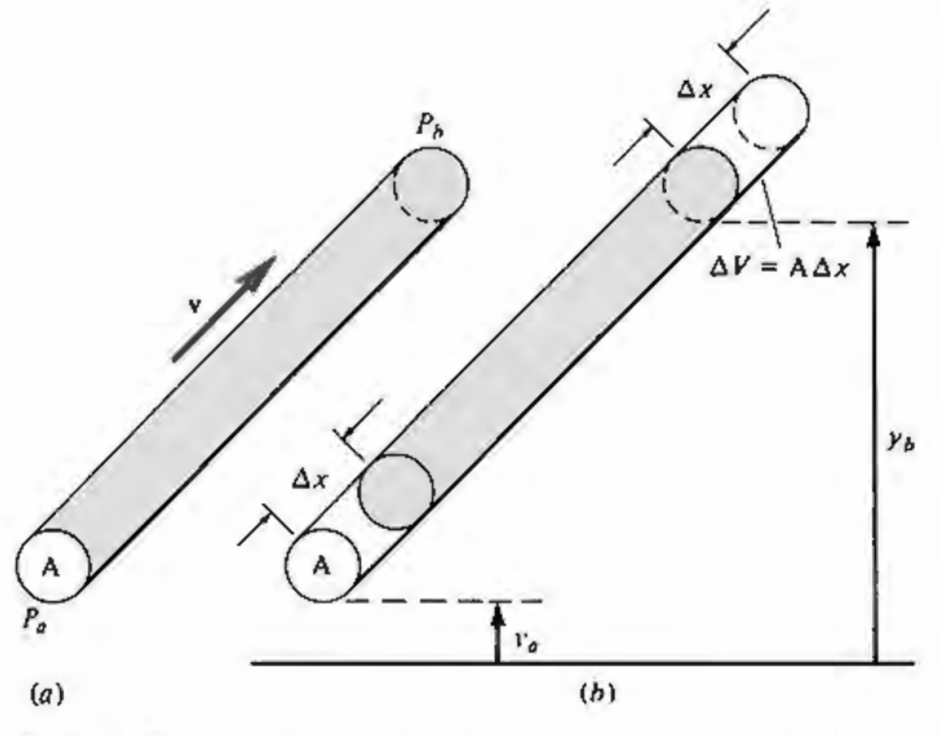
2. The flow rate is constant, so $A_1\ v_1 = A_2\ v_2$, and the velocity $v_2$ in the hose is

$$v_2 = v_1\frac{A_1}{A_2} = v_1\frac{\pi r_1^2}{\pi r_2^2} = v_1\left(\frac{r_1}{r_2}\right)^2 = (0.159\ \text{m s}^{-1})\frac{1}{(0.5)^2} = 0.636\ \text{m s}^{-1}$$

# 13.3 Bernoulli's equation

- **Bernoulli's equation** states the consequences of the principle that the work done on a fluid as it flows from one place to another is equal to the change in its mechanical energy.
- **Bernoulli's equation** can be used for the following conditions :
    1. The fluid is incompressible, then its density remains constant.
    2. The fluid is non-viscous (no mechanical energy is lost).
    3. The flow is streamline, not turbulent.
    4. The velocity of the fluid at any point does not change during the period of observation. (This is called the steady-state assumption.)

- Consider the fluid in a section of a flow tube with a constant cross section $A$ (Fig. a).
- According to the equation of continuity, the product $Av$ remains constant.
- Thus the velocity $v$ does not change as the fluid moves through the tube, and its kinetic energy remains the same.
- However, the potential energy changes as the fluid rises.



# 13.3 Bernoulli's equation

- The net force on the fluid in the tube due to the surrounding fluid is the cross-sectional area A times the difference in pressures on the ends, or

$$F = (P_a - P_b)A$$

- If the fluid in the section moves a short distance $\Delta x$, then the work done on it is the product of the force and the displacement

$$W = F\Delta x = (P_a - P_b)A\Delta x$$

- With $\Delta V = A\Delta x$, then

$$W = (P_a - P_b)\,\Delta V$$

- This work done on the fluid must equal the in crease $\Delta\mathcal{U}$ in its potential energy.
- $\Delta\mathcal{U}$ can be calculated if we note that the fluid leaving the section has a mass $\rho\Delta V$ and a potential energy $(\rho\Delta V)\mathrm{g}y_{\mathrm{b}}$, while the fluid entering at the bottom of the section has a potential energy $(\rho\Delta V)\mathrm{g}y_{\mathrm{a}}$. Thus,

$$\Delta\mathcal{U} = \rho g\,\Delta V(y_b - y_a)$$

- Equating this to $W$, we have

$$\boxed{P_a - P_b = \rho g(y_b - y_a)}$$

or

$$\boxed{P_a + \rho g y_a = P_b + \rho g y_b \qquad (v = \text{constant})}$$

Thus, the pressure $P$ plus the potential energy per unit volume $\rho g y$ of the fluid is the same everywhere in the flow tube if the velocity remains constant.

# 13.3 Bernoulli's equation

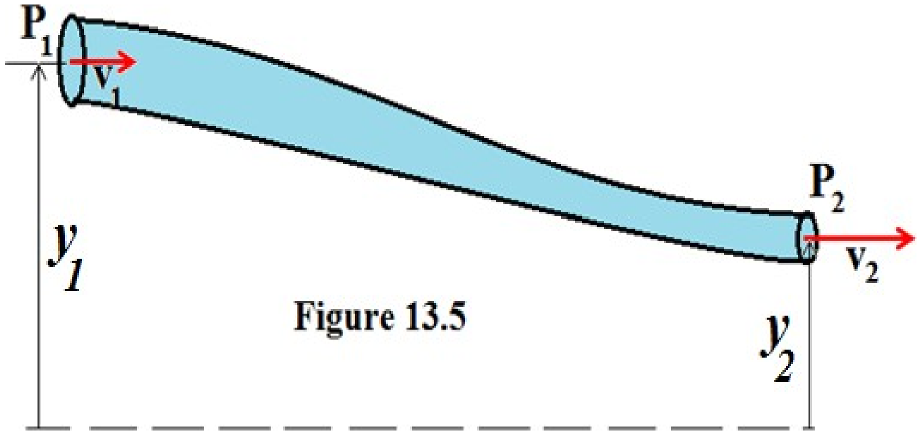


Figure 13.5

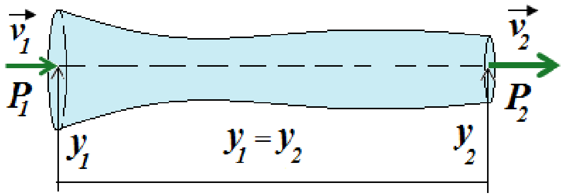
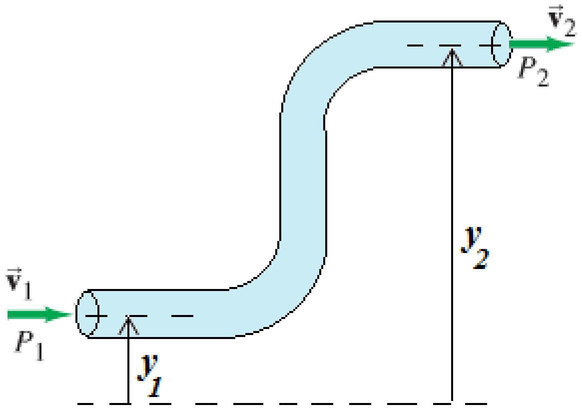
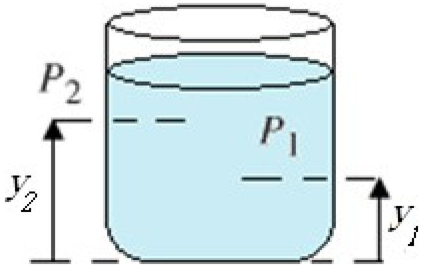
- More generally, if the cross-sectional area of the flow tube changes, the fluid velocity $v$ and kinetic energy per unit volume $\frac{1}{2}\rho v^2$ will also change.
- The work done on the fluid must then be set equal to the change in the potential plus kinetic energy of the fluid. The result is Bernoulli's equation,

$$\boxed{P_1 + \frac{1}{2}\rho {v_1}^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho {v_2}^2 + \rho g y_2}$$

- The pressure plus the total mechanical energy per unit volume, $P + \rho g y + \frac{1}{2}\rho v^2$, is the same everywhere in a flow tube.

# 13.3 Bernoulli's equation

## Specific forms of the Bernoulli's equation

| Case | Schematic representation | Bernoulli's Equation |
|---|---|---|
| Horizontal tube with non-uniforme size<br>$y_1 = y_2$ |  | $P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$ |
| Non-Horizontal tube with uniform size.<br>$A_1 = A_2$<br>$A_1 v_1 = A_2 v_2$<br>Then $v_1 = v_2$ |  | $P_1 + \rho g y_1 = P_2 + \rho g y_2$ |
| Static fluid $(\boldsymbol{v} = \mathbf{0})$ |  | $P_1 + \rho g y_1 = P_2 + \rho g y_2$<br>Hydrostatic Equation |

# 13.4 Static consequences of Bernoulli's equation

When the fluid is at rest ($\boldsymbol{v} = \mathbf{0}$), Bernoulli's equation is written as:

$$P + \rho g y = \ constant$$

**Pressure in a fluid at rest:**

the last form of the Bernoulli's equation can be used to calculate the pressure everywhere in the fluid. For example, from the figure find the pressure at a point $B$ in terms of the pressure at surface and the depth.

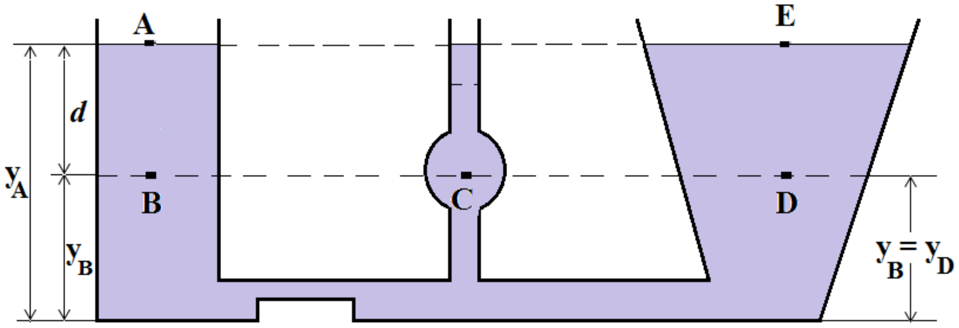



**Figure 13.6** Fluid at rest in a container. The level of the surface pf the fluid is the same in each segment

Using Bernoulli's equation, we can write:

At the surface $S_A$, $P_A + \rho g y_A = constant$ and at the surface $S_B$, $P_B + \rho g y_B = constant.$

Then

$$P_A + \rho g y_A = P_B + \rho g y_B$$

or

$$P_B = P_A + \rho g \ (y_A - y_B) = P_A + \rho g d$$

If the pressure at the surface $S_A$ is equal to the atmospheric pressure so $P_A = P_{atm}$

then :

$$P_B = P_{atm} + \rho g d$$

# 13.4 Static consequences of Bernoulli's equation

- This result shows that pressure at a depth $d$ in a fluid at rest is equal to the surface pressure plus the potential energy density change $\rho g d$ corresponding to this depth.

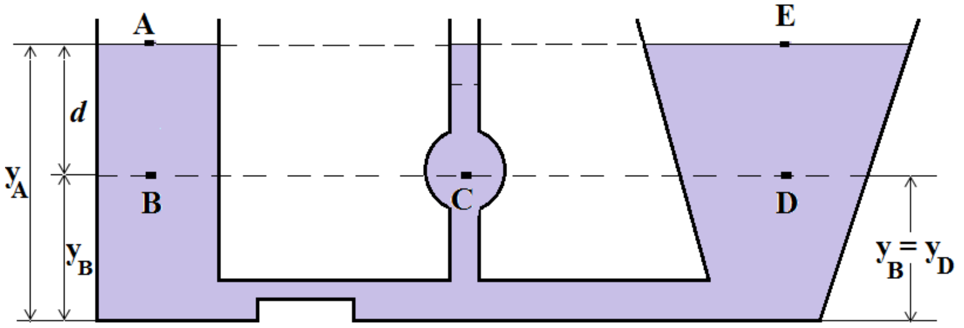



**Figure 13.6** Fluid at rest in a container. The level of the surface pf the fluid is the same in each segment

- Calculating $P + \rho g y$ at points $B$ and $D$ gives:

$$P_B + \rho g y_B = P_D + \rho g y_D$$

since $y_B = y_D$ then

$$\boxed{P_B = P_D}$$

- Thus, the pressure at the same depth at two places in a fluid at rest is the same. The surfaces of liquids at rest in connected containers of any shape must be at the same height if they are open to the atmosphere.

**Example 13 .5** :

What is the pressure on a swimmer 5 m below the surface of a lake?

**Answer**: Using $d = 5\ m$ and $\rho = 1000\ kg\ m^{-3}$, we find

$$\begin{aligned} P_B &= P_{\text{atm}} + \rho g d \\ &= 1.013 \times 10^5\ \text{Pa} + (1000\ \text{kg m}^{-3})(9.8\ \text{m s}^{-2})(5\ \text{m}) \\ &= 1.50 \times 10^5\ \text{Pa} \end{aligned}$$

# 13.4 Static consequences of Bernoulli's equation

**Example :** The pressure at the floor is measured to be normal atmospheric pressure, its value is $P_{atm} = 1.013\ bar$.

1) How much is the pressure at a height of $1000\ m$ .
2) You are in scuba diving at a $10\ m$ depth, you feel pain in the ears. Explain why?

*Solution:*

Here, $d = 1000\ m$. From Table 13.1 (p. 315) the density of air at atmospheric pressure and $0°\ C$ is $\rho = 1.29\ kg.m^{-3}$

Thus:

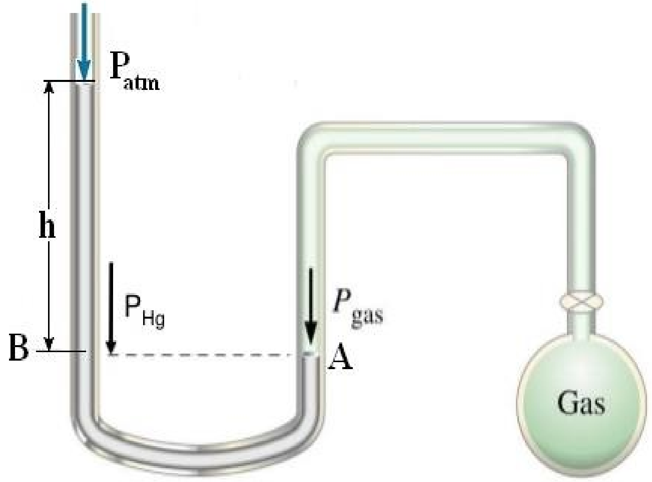
$$P_B = P_{atm} + \rho g d$$

$$= 1.013 \times 10^5\ Pa - (1.29\ kg.m^{-3}) \times (9.8\ m.s^{-2}) \times (1000m)$$

$$= 88.7\ kPa$$

# 13.4 Static consequences of Bernoulli's equation

**The manometer**: the open-tube manometer is a U-shaped tube used for measuring gas pressures (or liquid pressure if doesn't mix with the manometer fluid). It contains a liquid that may be mercury or, for measurements of low pressures, water or oil.
In the Figure the pressure of the gas ( *the pressure to measure*) is equal to the pressure on the liquid at the left arm $P_A = P_{gas}$
At the right arm the pressure of the mercury is $P_B = P_{atm} + \rho g h$
As $P_A = P_B$ (same level), then:




$$P_{gas} = P_{atm} + \rho g h$$

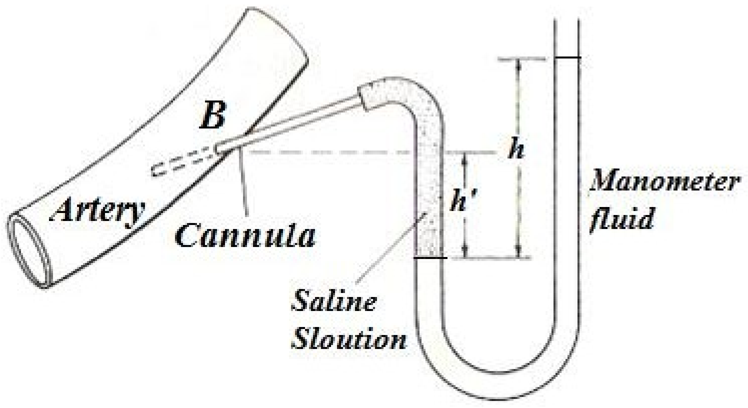
Thus, a measurement of the height difference h of the two columns determines the gas pressure $P_{gas}$

**The gauge pressure** : is the difference between the absolute pressure and the atmospheric pressure. In the above equation $P_{gas}$ is the absolute pressure, then the gauge pressure is $P_g = P_{gas} - P_{atm}$

$$\boxed{P_g = P_{gas} - P_{atm} = \rho g h}$$

For example, the blood pressure given by a sphygmomanometer is the gauge pressure $\rho g h$

# 13.4 Static consequences of Bernoulli's equation




**Cannulation**: In many experiments with anesthetized animals, the blood pressure in an artery or vein is measured by the direct insertion into the vessel of a cannula, which is a small glass or plastic tube containing saline solution plus an anticlotting agent.

The saline solution, in turn, is in contact with the fluid in a manometer. It's necessary to have the surface of contact between the saline solution and the manometer fluid either at the same level as the insertion point of the cannula or to correct for the height difference.

The pressure at the artery is:

$$P_B = P_{atm} + \rho g h - \rho_s g h'$$

Where $\rho_s$ is the density of the saline solution and $\rho$ is the density of the manometer fluid.

# 13.6 Blood pressure-Sphygmomanometer

During a complete heart pumping cycle, the pressure in the heart and the circulatory system goes through both a maximum *Systolic* ( as the blood is pumped from the heart) and a minimum *Diastolic* ( as the heart relaxes and fills with blood returned from the veins).
The sphygmomanometer is used to measure these extreme pressures.

The sphygmomanometer (Figure 13.10) is used in the upper human arm where it gives values nearly close to the pressure in the heart. Also, the upper arm contains a single bone making the brachial artery located there easy to compress.

Blood pressures are usually presented as $Systolic/Diastolic$ ratios.

Typical readings for resting healthy adult are about $120/80$ in torr (mm Hg), the borderline for high pressure (hypertension) is usually defined to be $140/90$. Pressures above that level needs medical attention.


**Figure 13.10** The sphygmomanometer